AMERICAN Journal of Engineering, Mechanics and Architecture
Volume 3, Issue 9, 2025 ISSN (E): 2993-2637

Application of the Fure Method in Studying Wave Propagation During the Movement of a Fluid in a Damper Pipe

Sh.Sh. Khojikulov, I.K. Khujaev
Institute of Mechanics and Seismic Stability of Structures named after M.T. Urazbayev, Uzbekistan

B.B. Bakhtiyorov, Kh.A. Mamadaliev
Tashkent University of Information Technologies named after Muhamad al-Khwarizmi, Uzbekistan

Abstract: An analytical solution to the linearized problem describing the propagation of pressure and fluid velocity disturbances in an inclined section of a pipeline with an air cap is obtained taking into account the gravitational effect, friction forces, and a local inertial component. The effect of the air cap is modeled based on I.A. Charny's approach. The initial equations formulated based on N.E. Zhukovsky's theory are solved using the method of separating variables in the presence of mixed boundary conditions. Based on the obtained solution, numerical modeling is performed, which makes it possible to identify the nature of changes in hydrodynamic disturbances depending on a number of factors, including the volume of the air cap.

Keywords: pipeline, incompressible fluid, damper, resistance force, gravity, quasi-one-dimensional model, pressure, Fure method, computational experiment.

Introduction

The processes of formation and propagation of disturbances in pipelines are studied on the basis of quasi-one-dimensional equations developed by N.E. Zhukovsky [1,2]. He was the first to formulate a system of quasi-one-dimensional equations that simultaneously takes into account both the hydrodynamic flow velocity and the propagation velocity of small disturbances in the medium-pipeline system. In addition, Zhukovsky conducted theoretical and experimental studies devoted to the propagation of compression and rarefaction waves in pipelines.

The development and wide practical application of pipeline networks is inextricably linked with in-depth theoretical research. Various mathematical models of pipeline transport of weakly compressible and super compressible media [3], as well as hydrodynamic mixtures within the framework of Newtonian and non-Newtonian fluids [4], covering both linear and nonlinear, as well as complete and simplified descriptions, have been developed. Analytical [5,6], numerical [7,8] and approximate methods for solving problems have been actively developed, both for the entire pipeline network and for its individual sections - with or without taking into account various force and energy factors.

In the model, the pipeline slope is assumed to be constant. In addition, the local component of the fluid inertia force and the resistance force according to the linearized Darcy-Weisbach formula are taken into account in the momentum conservation equation. The continuity equation is presented through the propagation velocity of small pressure disturbances in the pipe-fluid medium.

The linearized equation for the flow velocity is solved by the Fourier method. By substituting the found solution for the velocity problem, a transformed equation for conservation of mass is obtained, and by integrating it, a solution for the problem for hydrostatic pressure is obtained.

Numerical results of the process of propagation of compaction and rarefaction waves under conditions of damping by an air cap for individual variants of horizontal and inclined pipelines are presented and analyzed.

Statement of the problem

The objective of the problem is to study the dynamic state of an elementary section of a pipeline during the transition from one operating mode to another. A pipeline with a length l and diameter is considered D_0. The slope of the pipeline route is constant and is $\sin\alpha$. The initial condition for the velocity is taken to be

$$w(x,0) = w_0 = const.$$

The initial pressure distribution takes into account the input pressure p_{00}, the pressure drop due to friction and gravity:

$$p(x,0) = p_{00} - \rho(2aw_0 + g\sin\alpha)x.$$

Here $2a = \dfrac{\lambda w_*}{2D} = const$; λ – drag coefficient; w_* – characteristic speed of the object under consideration (in this case, the averaging parameter); $\sin\alpha = \dfrac{dy}{dx}$; $y(x)$ – leveling height of the pipeline axis.

The input pressure value is set

$$p(0,t) = p_{00} = const.$$

The intensity of liquid extraction from the end of the section is $t > 0$ $Q(t)$ $(м^3/c)$. An air cap is installed before the exit from the section. The volume and pressure of the gas in the air cap in an undisturbed state are V_0 and p_0. We formulate this boundary condition, reflecting the connection of the air cap, according to I.A. Charny [1].

Before the air cap, the volumetric flow rate of the liquid is $(fw)_{x=l}$, where $f = \pi D^2/4$ is the cross-sectional area, and D is the diameter of the pipeline. At the outlet, as already noted, the volumetric flow rate of the liquid is $Q(t)$. Their difference leads to a change in the volume of gas in the air cap over time:

$$\frac{dy}{dx} = (fw)_{x=l} - Q(t).$$

When changing the volume of the liquid, the change in the temperature of the gas can be neglected. Therefore, the new state of the air p and $V_0 - y$ satisfies the condition:

$$p_0 V_0 = p(V_0 - y).$$

That is, the new value of pressure in the air cap is

$$p = \frac{p_0 V_0}{V_0 - y}.$$

Because the change y is small $(y << V_0)$, then we can accept

$$p = \frac{p_0}{1 - \frac{y}{V_0}} \approx p_0\left(1 + \frac{y}{V_0}\right).$$

From here we find

$$y = \frac{p - p_0}{p_0} V_0 \text{ and } \frac{dy}{dt} = \frac{V_0}{p_0}\frac{dp}{dt}.$$

By equating the right-hand sides of the two equalities with respect to $\frac{dy}{dt}$, we obtain the condition at the exit from the section:

$$\frac{V_0}{p_0}\frac{dp_{x=l}}{dt} = (fw)_{x=l} - Q(t).$$

We model the equations of the state of the section based on the equations of N.E. Zhukovsky [1] with an amendment – taking into account the force of gravity:

$$\begin{cases} -\dfrac{\partial p}{\partial x} = \rho\left(\dfrac{\partial w}{\partial t} + 2aw + g\sin\alpha\right), \\ -\dfrac{\partial p}{\partial t} = \rho c^2 \dfrac{\partial w}{\partial x}. \end{cases}$$

Here $c = \left(\frac{\rho_0}{k} + \frac{D\rho_0}{E\delta}\right)^{-1/2}$ is the speed of propagation of small disturbances in the liquid-pipe system; ρ_0, k is the density of the liquid at rest and its modulus of elasticity; E, δ – Young's modulus of the material from which the pipeline is made and the thickness of the pipe $(\delta << D)$.

Since $\frac{\partial p}{\partial t}$ it can be expressed according to the second equation of the system, the second boundary condition of the problem for speed w takes the form:

$$-\beta\frac{\partial w(l,t)}{\partial t} = w(l,t) - w_A.$$

When the damper is disconnected, this condition becomes a condition of the first kind : $w(l,t) = w_A$. From here on, the notation is used

$$\beta = \frac{\rho c^2 V_0}{f p_0}.$$

In general, this formulation of the problem differs from other problems in that both the slope of the route and the presence of a damper at the end of the section are taken into account.

Analytical solution of the problem

For a constant value of the input pressure, we will single out the problem with respect to the speed. The following conditions are appropriate for it:

$$w(x,0) = w_{00}, \ \frac{\partial w(x,0)}{\partial t} = 0,$$

$$\frac{\partial w(0,t)}{\partial x} = 0, \ \beta \frac{\partial w(l,t)}{\partial x} + w(l,t) = w_A,$$

where we restrict ourselves to considering the case when the new velocity at the exit from the section is w_A. Here the boundary condition at $x = 0$ corresponds , according to the equation of conservation of mass, to the case $\frac{\partial p(0,t)}{\partial t} = 0$, i.e. $p(0,t) = p_{00}$.

In this case, a telegraph-type equation is formed from the original system [9]:

$$\frac{\partial^2 w}{\partial t^2} + 2a\frac{\partial w}{\partial t} = c^2 \frac{\partial^2 w}{\partial x^2}.$$

To apply the Fourier method, the boundary conditions of the problem must be brought to a homogeneous form. In our case, this is possible if we accept the replacement [9,10]

$$u(x,t) = w(x,t) - w_A.$$

In this case, the equations, initial conditions and the first boundary condition are written simply through $u(x,t)$, and the second boundary condition takes on a homogeneous form:

$$\frac{\partial u(l,t)}{\partial x} + \frac{1}{\beta} u(l,t) = 0.$$

We seek a solution $u(x,t)$ in the form:

$$u(x,t) = X(x)Y(t).$$

Then, according to the rules of the Fure method [9] , we have

$$\frac{Y''(t) + 2aY'(t)}{Y(t)} = \frac{X''(x)}{X(x)} = -\lambda^2.$$

Here $\lambda > 0$, otherwise we get a trivial (zero) solution to the problem.

Let's compose an autonomous equation for $X(x)$:

$$X''(x) + \lambda^2 X'(x) = 0.$$

We seek its solution in the form

$$X(x) = B\sin\lambda x + C\cos\lambda x.$$

The implementation of boundary conditions leads to particular eigenfunctions [9,10]:

$$X_n(x) = \cos\lambda_n x,$$

where are the eigenvalues λ_n. problems are positive roots of the characteristic equation

$$tg\lambda_n l = \frac{1}{\beta\lambda}.$$

Proved orthonormality of $X_n(x)$ eigenfunctions [9,10] :

$$\int_0^l \sin\lambda_n x \sin\lambda_m x dx = \begin{cases} \left\|X_n(x)\right\|^2 = \frac{1}{2}\left(l + \beta\sin^2\lambda_n l\right) \text{ при } n = m, \\ 0 \text{ при } n \neq m. \end{cases}$$

Finding eigenfunctions over time yielded the equation:

$$Y_n''(t) + 2aY_n'(t) + c^2\lambda_n^2 Y_n(t) = 0.$$

The characteristic equation of this second-order differential equation has the form:

$$s_n^2 + 2as_n + c^2\lambda_n^2 = 0.$$

When designating

$$D_n = a^2 - c^2\lambda_n^2$$

we will receive

$$(s_n)_{1,2} = -a \pm \sqrt{D_n}.$$

In this regard we have:

$$Y_n(t) = \begin{cases} e^{-at}\left(A_n ch\sqrt{D_n}t + B_n sh\sqrt{D_n}t\right) \text{ при } D_n > 0, \\ e^{-at}\left(A_n + B_n t\right) \text{ при } D_n = 0, \\ e^{-at}\left(A_n \cos\sqrt{|D_n|}t + B_n \sin\sqrt{|D_n|}t\right) \text{ при } D_n < 0. \end{cases}$$

Thus, the solution $u(x,t)$ is

$$u(x,t) = \sum_{n=1}^{\infty} \begin{bmatrix} e^{-at}\left(A_n ch\sqrt{D_n}t + B_n sh\sqrt{D_n}t\right) \text{ при } D_n > 0 \\ e^{-at}\left(A_n + B_n t\right) \text{ при } D_n = 0 \\ e^{-at}\left(A_n \cos\sqrt{|D_n|}t + B_n \sin\sqrt{|D_n|}t\right) \text{ при } D_n < 0 \end{bmatrix} \cos\lambda_n x.$$

In a particular case $\beta \to 0$ (i.e., when $V_0 \to 0$) taking into account the condition, $X_n'(0) = 0$ the eigenfunctions will be $X_n(x) = \cos\lambda_n x$ when $\lambda_n = \frac{2n-1}{2}\frac{\pi}{l}$ [5]. In this case , $\left\|X_n(l)\right\|^2 = \frac{l}{2}$.

Using the orthonormality of the eigenfunctions and according to the initial conditions, the values of the coefficients are determined

$$A_n = \frac{w_0 - w_A}{\lambda_n \|X_n\|^2} \sin \lambda_n l, \quad B_n = \frac{aA_n}{\gamma_n}.$$

Here $\gamma_n = \begin{cases} \sqrt{D_n} \ \text{при } D_n > 0, \\ 1 \ \text{при } D_n = 0, \\ \sqrt{|D_n|} \ \text{при } D_n < 0. \end{cases}$

The final solution to the problem regarding the flow velocity is:

$$w(x,t) = w_A + e^{-at} \sum_{n=1}^{\infty} A_n \begin{bmatrix} ch\sqrt{D_n}t + \frac{a}{\sqrt{D_n}} sh\sqrt{D_n}t \ \text{при } D_n > 0 \\ 1 + at \ \text{при } D_n = 0 \\ \cos\sqrt{|D_n|}t + \frac{a}{\sqrt{|D_n|}} \sin\sqrt{|D_n|}t \ \text{при } D_n < 0 \end{bmatrix} \cos \lambda_n x.$$

For $\beta \neq 0$ the values of the eigenvalues λ_n we find a numerical solution of the characteristic equation. First, we identified the boundary of the interval of membership n of the -th root : $\frac{(n-1)\pi}{l} < \lambda_n < \frac{(n-0,5)\pi}{l}$. Then the values $\lambda_n l$ were refined by dividing the segment in half [11]. In this case, the largest number of approximation steps 42 was sufficient to ensure the accuracy of the calculation λ_n up to 10^{-10} at $l = 1000$ m.

To find the hydrostatic pressure, the second equation of the original system was integrated over time from 0 to t:

$$p(x,t) = p(x,0) - \rho c^2 \int_0^t \frac{\partial w(x,\theta)}{\partial x} d\theta.$$

The minuend is known from the initial condition. The subtrahend is calculated using the newly obtained expression for $w(x,t)$. Omitting the details, we present the final result:

$$p(x,t) = p_{00} - \rho(2aw_0 + g \sin \alpha)x -$$

$$-\rho \sum_{n=1}^{\infty} \frac{A_n}{\lambda_n} \begin{bmatrix} -2a\left(e^{-at} ch\sqrt{D_n}t - 1\right) - \left(\gamma_n + \frac{a^2}{\gamma_n}\right) e^{-at} sh\sqrt{D_n}t \ \text{при } D_n > 0 \\ -2a\left(e^{-at} - 1\right) + a^2 te^{-at} \ \text{при } D_n = 0 \\ -2a\left(e^{-at} \cos\sqrt{|D_n|}t - 1\right) + \left(\gamma_n - \frac{a^2}{\gamma_n}\right) e^{-at} \sin\sqrt{|D_n|}t \ \text{при } D_n < 0 \end{bmatrix} \sin \lambda_n x.$$

Discussion of results

Based on the presented material, a calculation program was compiled in the Pascal ABC environment, where the results were presented in the form of tables. Graphs were constructed using the Excel program . The characteristic equation was solved using the dichotomy method with an accuracy of 10^{-10}. The first 1000 terms of the Fourier series were taken into account in the calculations. The step along the length of the section was $l/100$, and the step along the time was $l/(10c)$. The calculation was carried out from the 0th to the 600th time step.

Cases $\sin\alpha = 0,\ \pm 0.1$ with a section length of 1000 m were considered . The section diameter was 20 cm, and the resistance coefficient $\lambda = 0.018$ was . The averaging parameter had a value of $w_* = 5\ м/с$, the density of the liquid in the unperturbed state was $1000.0\,кг/м^3$, and the propagation speed of small pressure disturbances was $c = 1200\ м/с$.

The volume of the air cap connected to the end of the section was taken as 1.0, 0.100, 0.010, 0.001, 0.0001 and 0.00001 cubic meters. The pressure in the air cap without voltage was taken as $p_0 = 0.1\,МПа$.

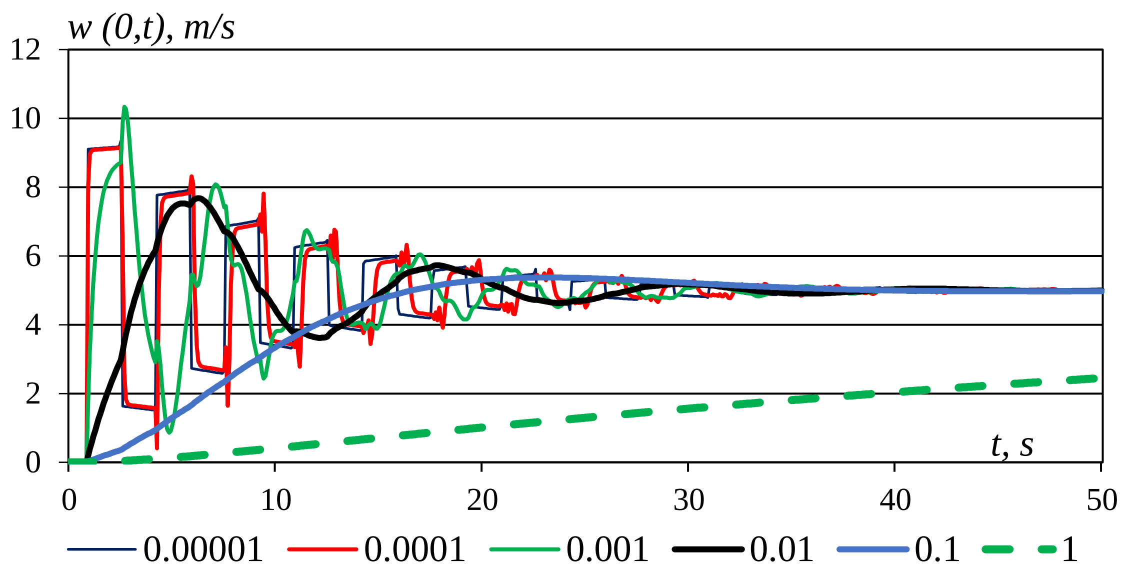

Fig. 1. Temporal change in inlet velocity for different values of air cap volume V_0. $l = 1000\,м$, $D = 0.20\,м$, $p_{00} = 6.5\,МПа$, $w_A = 5\,м/с$, $\lambda = 0.018$, $c = 1200\,м/с$, $p_0 = 0.1\,МПа$, $\sin\alpha = 0$

Fig. 1 and 2 show the changes in the velocity value at the ends of the section for different values of the air cap volume. The lower graphs in them reflect the end velocities at $V_0 = 1.000\ м^3$. These graphs are not complete, since the flow velocity subsequently passes to its limiting value w_A. That is, at large volumes of the air cap, the establishment occurs slowly, but without oscillations. It is evident that at the end of the section where the air cap is installed, the amplitude of the velocity disturbances is lower than in the inlet section. It is evident that the end velocities increase monotonically. The graphs at $V_0 = 0.1\ м^3$ first increase from 0 to 5 $м/с$, and then the oscillatory process is damped. Two conditional maxima are clearly visible in the graphs. More than three conditional maxima were observed in the graphs at $V_0 = 0.00100\ м^3$.

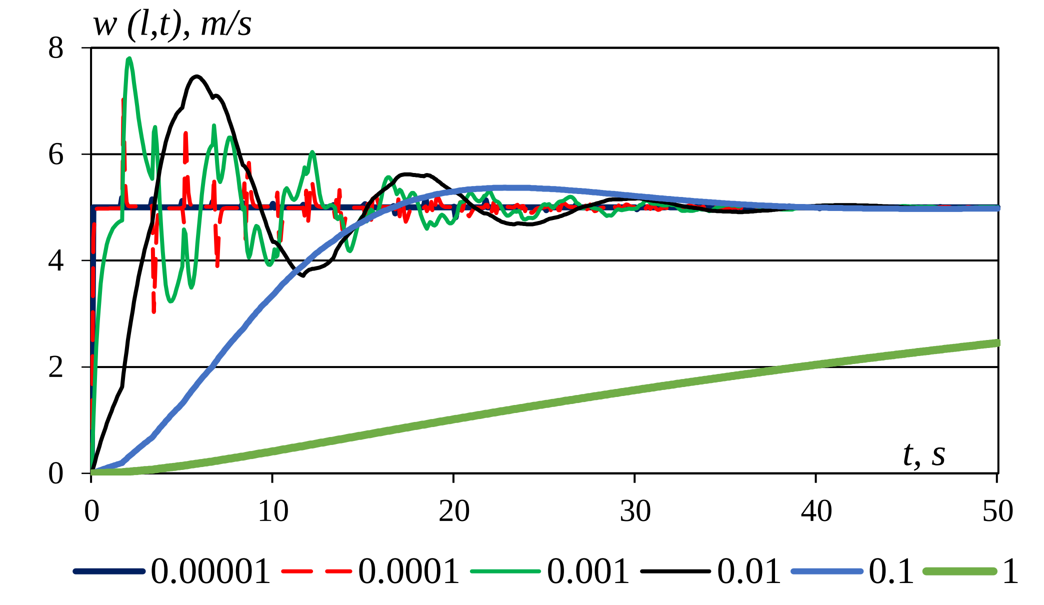

Fig. 2. Temporal change in the output velocity at different values of the air cap volume V_0

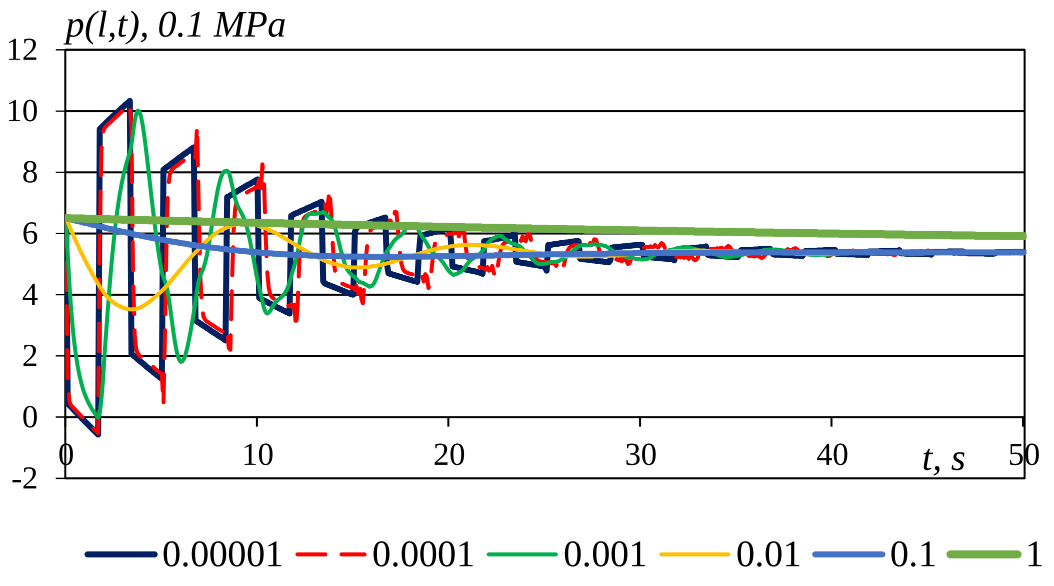

Fig. 3. Temporary change in outlet pressure at different values of air cap volume V_0

At smaller volumes of the air cap, the frequency of disturbances corresponds to the frequency of disturbances when the air cap is switched off. In general, when the volume of the damper decreases, the amplitude of the velocity disturbances at the ends of the section increases, and the frequency of disturbances increases.

Let's move on to the analysis of the results obtained with positive $(\sin\alpha = 0.100)$ and negative $(\sin\alpha = -0.100)$ slopes of the route.

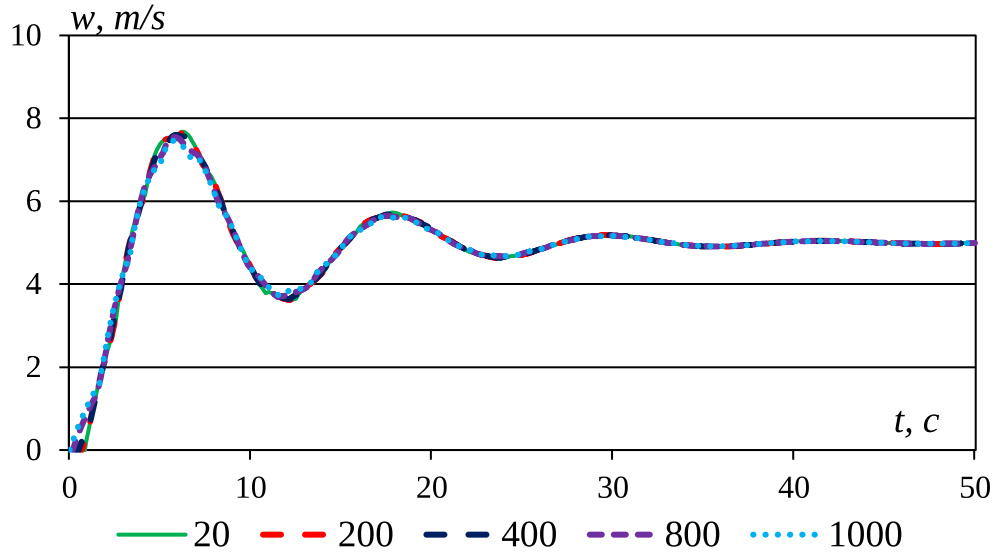

Fig. 4. Temporal changes in flow velocity in different sections of a linear section at $V_0 = 0.010\ м^3 . \sin\alpha = 0.100$

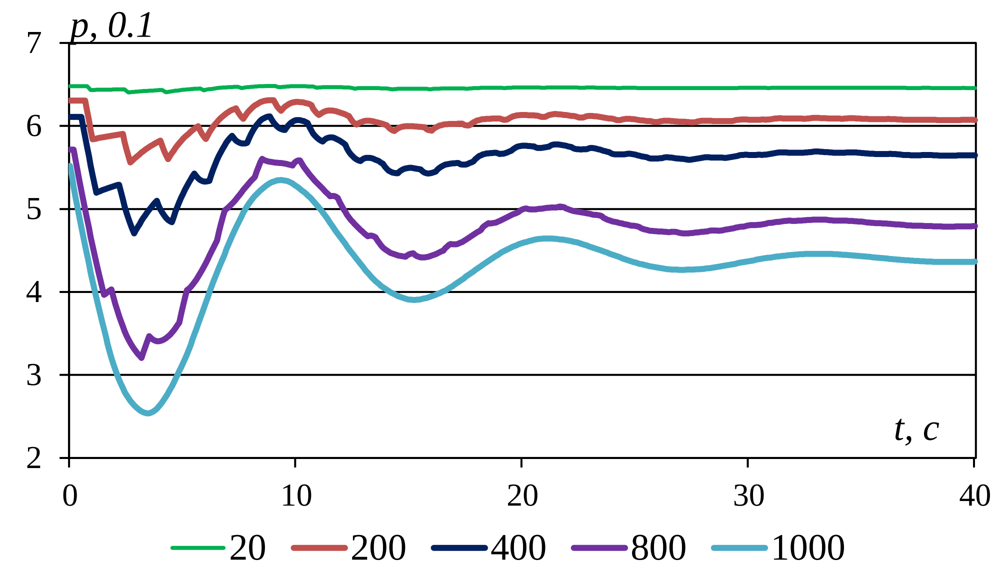

Fig. 5. Temporal changes in pressure in different sections of a linear section at $V_0 = 0.010\ м^3 .$ $\sin\alpha = 0.100$

Fig. 4 shows graphs of the time change in flow velocity in different sections with $V_0 = 0.010\ м^3$ and maintaining the values of the remaining initial data. As can be seen from the figure, minor differences in the results are observed around the primary velocity extremes.

The graphs of the time change of pressure in the sections corresponding to this calculation option are shown in Fig. 5. The fluctuations (oscillations) of the pressure relative to the average pressure value become noticeable with distance from the end of the section with the air cap. The second feature of the graphs is the decrease in the pressure value in the sections depending on the distance from the input section: an increase in distance leads to a greater pressure loss. In this case, such a nature of the pressure change is also due to the positive slope of the pipeline route, since a slope of was adopted $\sin\alpha = 0.100$.

Conclusions

Taking into account the air cap included at the end of the section, a mathematical model of the state of an elementary inclined and horizontal section of the pipeline was compiled, when a constant pressure value is set at the inlet to the section, and a flow velocity at the outlet. In this case, quasi-one-dimensional equations of conservation of momentum and mass are adopted according to the model of N.E. Zhukovsky, taking into account the slope of the route, and the condition for the damper is according to I.A. Charny [1].

The Fourier method was used to obtain a solution to the problem of flow velocity, and the solution to the pressure dynamics was found by integrating the mass conservation equation.

The results of calculations obtained for horizontal and inclined sections for different values of the air cap volume are discussed. It is revealed that at small values of the air cap volume, velocity and pressure disturbances are formed, the frequencies of which coincide with the frequencies of the solution of the problem without a damper. And at a large volume of the air cap, the transition from one mode to another operating mode occurs smoothly.

The results showed that as the end of the air-bubble section was approached, the velocity and pressure oscillations decreased.

It has been established that the frequency and amplitude of disturbances decrease with increasing volume of the air cap. With large volumes of the air cap, the settling time increases.

Literatures

1. Charny I.A. Unsteady motion of real fluid in pipes. 2nd ed. – M.: Nedra, 1975. – 296 p.,
2. Seleznev V.E., Aleshin V.V., Pryalov S.N. Modern computer simulators in pipeline transport. Mathematical methods of modeling and practical application / Ed. V.E.Seleznev . – M.: MAKS Press, 2007. – 200 p.
3. Charny , Korotaev Yu.P., Shirkovsky A.I. Extraction, transport and underground storage of gas. – M.: Nedra, 1997. – 487 p.
4. Fayzullaev D.F., Mukuk K.V. Sequential movements of liquids. – Tashkent: Fan, 1976. – 184 p.
5. Grachev V.V., Shcherbakov S.G., Yakovlev E.I. Dynamics of pipeline systems. – M.: Nauka, 1987. – 438 p.
6. Grachev V.V., Huseynzade M.A., Ksendz B.I., Yakovlev E.I. Complex pipeline systems. – M.: Nedra, 1982. – 256 p.
7. Atena A., Tekalign W., Muche T. Steady State Gas Flow in Pipeline Networks: Existence and Uniqueness of Solution // Journal of Applied Mathematics and Physics, (2020) 8 , 1155-1167. doi : 10.4236/jamp.2020.86087
8. Khujaev I.K., Bozorov O.Sh. , Mamadaliev Kh.A. , Aminov H.Kh. , Akhmadjonov SS Finite-difference method for solving nonlinear equations of traveling waves in main gas pipelines // Problems computing And applied mathematics . – Tashkent , 2020, No. 5(29). – P. 95-107 .
9. Tikhonov A.N., Samarsky A.A. Equations of mathematical physics. – M.: Nauka, 1977. – 736 p.
10. Khozhaev I.K., Khozhukov Sh.Sh., Shamsutdinova N.Sh. The Sturm- Liouville problem for an arbitrary combination of inhomogeneous boundary conditions of the first, second and third kinds of boundary conditions / / Problems of Computational and Applied Mathematics, 2023, No. 2 (47). - P. 125-128.
11. Kalitkin N.N. Numerical methods – M .: Science, 1978. – 512 p.